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Expanding and forwarding

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Abstract

Expanding parameters of graphs (magnification constant, edge and vertex cutset expansion) are related by very simple inequalities to forwarding parameters (edge and vertex forwarding indices). This shows that certain graphs have eccentricity close to the diameter. Connections between the forwarding indices and algebraic parameters like the smallest eigenvalue of the Laplacian or the genus of the graph are made. Graphs with unknown spectrum (de Bruijn, Kautz) are shown to be reasonable expanding by purely combinatorial arguments. Conversely, near-optimal routings in these graphs yield tight bounds on the spectrum.

1. Introduction

Expanding parameters of graphs have been studied by algebraic techniques (spectrum of the Laplacian, Harmonic Analysis [1, 2, 19, 20]) in order to construct explicit superconcentrators and by probabilistic techniques [16, 21, 9] to bound the rate of convergence of random walks on graphs. So far the vertex and edge forwarding index, two parameters characterizing the congestion of an interconnection network [7, 15, 23, 10] has been studied by combinatorial techniques only. In this article a connection is established between the expanding and forwarding parameters of a graph. Intuitively, this is an instance of the cut-flow duality, expanding parameters being measures of connectivity and forwarding indices being statistics on paths.

The paper is split in two parts: the first part deals with edge parameters, the second deals with vertex parameters. The reason for partitioning the paper in that way and in that order is that the duality between vertices and edges is only approximate: indeed edge results are easier to come by and more complete. After recalling definitions in Section 2.1 (Section 3.1), we develop our main results in Section 2.2 (resp. 3.2), then pause for some examples in Section 2.3 (resp. 3.3). We consider relations between the expanding parameters and the other possible definitions in Section 2.4 (resp. 3.4). Then, we give some more complex examples in Section 2.5 (resp. 3.5).

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2. Edge parameters

2.1. Notations and definitions

All graphs are undirected with vertex set V of cardinality v, edge set E of cardinality e, E being a subset of the set of all pairs of elements of V. Let $X \subseteq V$ be a proper subset of V. Let $\bar{X}$ denote the complement of X in V. The *edge-cut* ∂X induced by X and $\bar{X}$ is

$$\partial X = \{\{x, y\}/E,\ x \in X,\ y \in \bar{X}\}.$$

The *edge expanding factor* (called-up to a multiplicative factor – "strong cutset expansion" in [18]) is defined by

$$\beta = \min\left\{\frac{|\partial X|}{|X||\bar{X}|}\,\middle|\, X \subseteq V,\ 1 \leqslant |X| \leqslant v - 1\right\}.$$

An edge-cut where this minimum is met with equality is called *optimal.* A *routing* R of the graph Γ is a set of $v(v-1)$ paths connecting each ordered pair of distinct vertices of V. The load of an edge $d \in E$ in R is the number of paths $R(d)$ of R passing through it. The edge forwarding index of Γ is defined as in [15] as

$$\pi = \min_{R} \max_{d \in E} R(d).$$

A routing where this minimum is met with equality is called *optimal.* If each edge receives the same load the routing is called *uniform.*

The *eccentricity* in a graph is

$$\bar{D} = \frac{1}{v(v-1)} \sum_{(x,y) \in V^2} d(x, y),$$

where $d(u, v)$ is the length of a shortest path from u to v.

2.2. Basic connection

Here we elaborate on a principle described in [16, 21].

Theorem 1. *In a connected graph $\pi\beta \geqslant 2$ with equality only if there is an optimal routing which loads uniformly the edges of some cut with load π.*

Proof. Let R be an optimal routing. The total load induced by R over the edges of ∂X is at least $2|X||\bar{X}|$. By definition of π and the optimality of R this load is at most $\pi|\partial X|$. Hence for any X we have the bound

$$2|\bar{X}||\bar{X}| \leqslant \pi|\partial X|,$$

with equality only if each edge of ∂X has the load π in R. $\square$

This result shows that a "good" forwarder (small π) is a good expander (large β), that a bad expander is a bad forwarder, but not that a good expander is a good forwarder. An instance when the forwarding index is known is the following.

Theorem 2. *If there is an optimal routing consisting of shortest paths only and loading uniformly the edges then* $\beta \geqslant \beta_0$, *where* $\beta_0 = 2e/(v(v-1)\bar{D})$. *In particular if the graph is regular of degree* Δ, *then* $\beta_0 = \Delta/((v-1)\bar{D})$.

Proof. An uniform optimal routing loads all edges with load π, in particular the edges of some cut. The value of β_0 follows from the value of π in Theorem 3.2 of [15]. □

2.3. Elementary examples

The following three examples possess optimal uniform routing of shortest paths [15], so that Theorem 2 applies. A path of length $v-1$ has apparently – cut the path in the middle – edge expanding factor $\beta = (\lfloor v/2 \rfloor \lceil v/2 \rceil)^{-1}$ consistently with the value of $\pi = 2\lfloor v/2 \rfloor \lceil v/2 \rceil$ found in [16]. Analogously, for a cycle of length v, the minimum is obtained by cutting two antipodal edges yielding twice the preceding value for β consistently with the value of $\pi = \lfloor v^2/4 \rfloor$ found in [15]. For the n-dimensional hypercube the optimal cut is a perfect matching between two $n-1$-dimensional hypercubes, yielding $\beta = 2^{-n+1}$ consistently with the value of $\pi = 2^n$ found in [15]. This improves on the estimate 2^{-n} given in [9].

2.4. Other expanding parameters

In [19], working by analogy with Riemannian manifolds, Mohar calls *isoperimetric number* the following invariant

$$i = \min\left\{ \frac{|\partial X|}{|X|} \,\middle|\, X \subseteq V,\ 1 \leqslant |X| \leqslant \frac{v}{2} \right\}.$$

It is immediate from the definition that $i \geqslant \beta v/2$. From this simple remark, and application of Theorem 1, we obtain the following result. (Recall that the Laplacian L of a graph with adjacency matrix A and degree function d is the $v \times v$ matrix with entries $d_x \delta_{x,y} - A_{x,y}$.)

We refer to [19, 20] for more details on the Laplacian of a graph.

Theorem 3. *If the smallest nonzero eigenvalue of the Laplacian is* λ *and the maximum degree* Δ,

$$\pi \geqslant \frac{v}{\sqrt{v(2\Delta - \lambda)}}.$$

Proof. From Theorem 4.2 of [19]. □

This is the first result connecting the edge-forwarding index with the spectrum of the Laplacian. Applications of the following result will be given in the next subsection.

Theorem 4. *In a graph with v vertices and maximum degree Δ the diameter is bounded from above by*

$$2\left\lceil \frac{\ln(v/2)}{\ln((\pi\Delta + v)/(\pi\Delta - v))} \right\rceil.$$

Proof. From Corollary 2.4. of [19]. □

This theorem means that certain graphs have an eccentricity close to the diameter.

Corollary 1. *In a regular graph of degree Δ which affords an optimal routing of shortest paths the diameter D is such that*

$$D \leqslant 2\left\lceil \bar{D} \ln\left(\frac{v}{2}\right) \right\rceil.$$

Proof. From Theorem 3.2 of [15], we know that $\pi = (v(v-1)/e)\,\bar{D}$. Since the graph is regular $2e = v\Delta$, entailing $\pi\Delta = 2(v-1)\bar{D}$. The result follows from Theorem 4 and the inequality valid for $0 < x < 1$,

$$\ln(1+x) - \ln(1-x) \geqslant 2x. \qquad \square$$

We conclude by an unexpected connection between the edge forwarding index and the genus of the graph.

Theorem 5. *In a graph with v vertices, maximum degree Δ, and genus g such that $v > 18(g+2)^2$ we have*

$$\pi \geqslant \frac{v[\sqrt{v/2} - 3(g+2)]}{3\Delta(g+2)}.$$

Proof. From Proposition 7.1 of [19]. □

From there, we see that, for fixed genus the forwarding index cannot be too small. This means that low genus embedding is an obstruction to good routing.

2.5. Advanced examples

2.5.1. Distance-transitive graphs

In [15], it is conjectured (Conjecture 3.9) that for every distance-transitive graph $\pi = \lceil 2/\beta_0 \rceil$. We shall call this conjecture the HMS conjecture and will provide a family of examples where the HMS conjecture, if true, yields a very close approximation for β. Let $J(n,k)$ denote the Johnson graph on $v = \binom{n}{k}$ vertices [17, p. 665]. Recall that V is the set of all k-subsets of an n-set ($n > k$), two subsets being connected if they have exactly $k - 1$ elements in common. This graph is distance-transitive with diameter k and degree $k(n - k)$.

We claim that $\beta \leqslant (n - k)/\binom{n-1}{k}$. Let $X \subseteq V$ be the set of k-subsets that contain a given element of the groundset. Then $|\bar{X}| = \binom{n-1}{k}$, and every vertex of X has $n - k$ neighbors in $|\bar{X}|$. Hence $|\partial X| = (n - k)|X|$, and the expansion ratio for this special cut has the value

$$\frac{n-k}{\binom{n-1}{k}} = \frac{n}{\binom{n}{k}}.$$

It can be shown, by algebraic techniques [25], that $\bar{D} = k(1 - k/n)\, v/(v - 1)$, yielding, with the notations of Theorem 2 the lower bound

$$\beta_0 = \frac{2}{\lceil 2\binom{n}{k}/n \rceil},$$

which is the same, up to a ceiling function, as the preceding value.

Open Problem 1. *Find an optimal routing in $J(n,k)$.*

Using the fact that $\lambda_1(J(2d,d)) = d$ (the spectrum of $J(2d,d)$ is in [17, p. 665]) and the bound $\beta \geqslant \lambda_1/v$ from [19] one can show that $\beta \geqslant d/v$.

Note that this estimate improves on [9] where the weaker bound $\beta \geqslant 1/dv$ is derived by combinatorial arguments.

Open Problem 2. *Do optimal cuts in known distance-regular graphs provide counter-examples to the HMS conjecture?*

2.5.2. Graphs on alphabets

The best forwarding graph in [16] is the de Bruijn graph on $v = d^D$ vertices, with degree $2d$ and diameter D, which satisfies to $\pi \leqslant (2v/d)(D - 1)$. Surprisingly, this value is obtained for routings which are not shortest paths and are not symmetric. This entails

$$\beta \geqslant \frac{d}{v(D-1)}.$$

Open Problem 3. *Find the exact values of π and β for the de Bruijn graph.*

Analogous results and questions hold for the Kautz graph. Note that computing the Laplacian of the Kautz or the de Bruijn graph is so far an open problem[1] so that Theorem 3 is not combinatorially useful yet. However, we can use it to derive an

[1] Note: the spectrum of the de Bruijn graph has been computed by Delorme–Tillich (to appear).

estimate on the range of λ. We get

$$\lambda \in [\Delta - \sqrt{\Delta - v^2}, \Delta + \sqrt{\Delta - v^2}],$$

where v stands for the ratio $v = \Delta/4(D-1)$.

2.5.3. Orbital regular graphs

An interesting class of graphs satisfying the hypotheses of Theorem 2 is the class of *orbital regular graphs* [23, 10]. A graph is said to be orbital regular if it possesses a subgroup H of its automorphism group which acts regularly on the edges and on each of its non-trivial orbits on V^2. An example is the Waring graph with $V = \boldsymbol{F}_q$, the finite field with q elements and $E = \{(x, y) \in V^2 \mid \exists z \in \boldsymbol{F}_q,\ z \neq 0,\ x - y = z^m\}$. If d is even, this graph is undirected. For $m = 2$, we obtain the Paley graph P_q [4]. It is shown in [23] that $\pi(P_q) = 6$ with an optimal uniform routing of shortest paths. This yields $\beta(P_q) = \frac{1}{3}$.

2.5.4. Cyclic codes

We assume here some familiarity with cyclic codes and covering radius, and refer to [14, 17] for basic definitions. Recall that a graph can be defined with vertices the cosets of a projective $[n,k]R$ cover over $\boldsymbol{F}_q$ such that two cosets are connected if their difference is a coset of weight 1. Such a graph has diameter $D = R$, and constant degree $\Delta = n(q-1)$. In this case, $\bar{D}$ can be interpreted as the "gap" [24] of the code which measures the average rate of distortion when the code is used for source coding [3]. The next result means that for certain codes the coset graph is orbital regular, implying from Theorem 4 and its corollary some closeness between the values of the covering radius and the gap.

Theorem 6. *Let C be a cyclic $[n,k]$ code without words of period $< n$. Then its coset graph is orbital regular.*

Proof. Here H is the subgroup generated by the shift. Let $S \subseteq H$. The coset $x + C$ is a fixed point of S iff $S(x) - x \in C$. In the algebra $\boldsymbol{F}_q[y]/(y^n - 1)$ S acts by multiplication by a power of y, say m. Assume C is the ideal generated by $g(y)$. Then $x(y) + C$ is a fixed point of S iff $g(y)$ divides $(y^m - 1)x(y)$. By hypothesis $g(y)$ is coprime with $y^m - 1$. Hence $g(y)$ divides $x(y)$, which says that $x \in C$. $\square$

3. Vertex parameters

3.1. Notations and definitions

As in Section 2, X is a proper subset of V. The vertex cut induced by X is

$$N(X) = \{y \in V - X \mid \{x, y\} \in E\}.$$

Here X^+ denotes the complement of $X \cup N(X)$ in V. The *vertex expanding factor* is defined by

$$\gamma = \min\left\{\frac{|N(X)|}{|X||X^+|}\,\middle|\, X \subseteq V,\ 1 \leqslant |X| \leqslant (v-1),\ |X^+| \geqslant 1\right\},$$

where the min on a void set of X is taken to be infinite.

The *load* $R(x)$ of $x \in V$ is the number of paths of R admitting x as *inner* point. The *vertex forwarding index* ξ is defined by

$$\xi = \min_R \max_{x \in V} R(x).$$

A routing that meets this minimum is called *optimal*. A path that has an endpoint in X, the other in X^+ is called *transversal*.

3.2. Basic connection

Theorem 7. *In a connected graph, $\gamma\xi \geqslant 2$ with equality only if there is an optimal uniform routing which loads all the vertices of some vertex cut with load ξ, by using transversal paths only.*

Proof. analogous to the proof of Theorem 1 with the important difference that non-transversal paths are not accounted for. In symbols:

$$2|X||X^+| \leqslant \xi N(X). \qquad \square$$

Corollary 2. *If there is an optimal routing of shortest paths then*

$$\gamma(v-1)(\bar{D}-1) \geqslant 2.$$

Proof. By a theorem of Chung et al. [7], we know that then $\xi = (v-1)(\bar{D}-1)$. The proof follows on applying the preceding theorem. $\square$

Open Problem 4. *Find sufficient conditions for equality in the preceding theorem.*

3.3. Elementary examples

It is easy to see that equality holds in the preceding Theorem for the path and the cycle since then the optimal cuts contain less than 2 vertices. This yields $\xi(P_v) = 2\lfloor v/2 \rfloor(\lceil v/2 \rceil - 1)$ as in [7], and $\gamma(C_v) = 2(\lfloor v/2 \rfloor \lceil v/2 \rceil)^{-1}$ consistently with $\xi(C_v) = \lfloor (v-2)^2/4 \rfloor$. In contrast with β the exact value of γ for the n-cube is not known; from [15], we only known that $\gamma \geqslant 2((n-2)2^{n-1}+1)^{-1}$. Taking as X the Hamming ball of radius $n/2$ yields after straightforward calculations

$\gamma = O(1/(\sqrt{n}2^n))$.[2] It can be checked by computer that $\gamma = \frac{3}{4}$ for $n = 3$ and $\gamma = \frac{6}{25}$ for $n = 4$, corresponding to Hamming balls of radius 0 and 1, respectively. So the exact order of γ is still unknown. This fact shows that vertex parameters are harder to determine.

3.4. Other expanding parameters

Recall [1,2] that a (n, d, c) expander is a bipartite graph on $2n$ vertices with equal parts, constant degree $2d$, and such that for every X in the left part, say, of cardinality less than $n/2$

$$|N(X)| \geqslant |X| + c|X|\left(1 - \frac{|X|}{v}\right).$$

It is easy to check that for such a graph $c \geqslant 2n\gamma/(1 + n\gamma)$. On the other hand, from the knowledge of γ, nothing can be deduced about the existence of a possible factor c, because of the additional term in $|X|$ in the definition. By using the argument in the proof of [1, Corollary 2.3] we obtain the following bound.

Theorem 8. *In a graph on v vertices and of maximum degree Δ and smallest nonzero eigenvalue of the Laplacian λ the vertex expanding factor is*

$$\gamma \geqslant \frac{4\lambda}{v\Delta}.$$

For instance, for the n-cube we find $\gamma \geqslant 1/(n2^{n-3})$ improving on the estimate of Section 3.3 by a factor of 2. Another expanding coefficient of interest is the *magnification constant* [1] (also called cutset expansion in [18]) m defined as

$$m = \min\left\{\frac{|N(X)|}{|X|}\,\middle|\, X \subseteq V,\ 1 \leqslant |X| \leqslant \frac{v}{2}\right\}.$$

Theorem 9. *A graph with v vertices and forwarding index ξ has magnification constant $m \geqslant v/(v + \xi)$.*

Proof. By definition of γ, noticing that $|X^+| = v - |N(X)| - |X|$, we have

$$|N(X)| \geqslant \gamma|X|(v - |X|)/(1 + \gamma|X|),$$

and in particular for $|X| \leqslant v/2$, we get

$$|N(X)| \geqslant \gamma|X|\frac{v}{2}\bigg/\left(1 + \gamma\frac{v}{2}\right).$$

From $\gamma\xi \geqslant 2$, the result follows. □

[2] Note: this has been shown to be tight by I. Vrto (to appear).

Corollary 3. *A graph with v vertices and eccentricity $\bar{D}$ which affords an optimal routing of shortest paths has magnification constant*

$$m \geqslant \frac{v}{v + (v-1)(\bar{D}-1)} \geqslant \frac{1}{\bar{D}}.$$

In particular this holds for Cayley graphs.

Proof. The assertion on Cayley graph follows from [15, Theorem 3.6]. □

Due to the obvious inequality $m \leqslant i$ we obtain immediately the vertex analogues of Theorems 3, 4, 5 on replacing π by ξ. Again this is the first time that the first eigenvalue of the Laplacian and the genus appear in connection with the vertex forwarding index. Actually, we can obtain a slightly tighter estimate than both vertex and edge versions of Theorem 5 by using "separator theorems" which are hidden in the reference given in the proof of Theorem 5.

Theorem 10. *Let Γ be a graph of genus g on v vertices, and with maximum degree Δ. Then*

$$\xi = \Omega(v^{3/2}/\sqrt{g}),$$

and moreover,

$$\pi = \Omega(v^{3/2}/(\Delta\sqrt{g})).$$

Proof. From [13] we know that there is a partition of V into three sets A, B, C such that any path from A to C or from C to A goes through B, and such that $|B| = \mathrm{O}(\sqrt{vg})$ and both A and C are of cardinality $\leqslant 2v/3$. But since $|A| + |B| + |C| = v$, we see that both A and C are of cardinality $\geqslant v/3 + \mathrm{o}(v)$. Hence, letting $X = A$, $N(X) \subseteq B$ and $C \subseteq X^+$, we see that $\gamma = \mathrm{O}(\sqrt{g}/v^{3/2})$. From $\gamma\xi \geqslant 2$, the result on ξ follows. The result on π follows from the inequality

$$2\xi + v - 1 \leqslant \Delta\pi,$$

which is (i) of Proposition 2.1 of [15]. □

3.5. Advanced examples

3.5.1. Vertex-transitive graphs

In [14] it is conjectured (Conjecture 3.5) that in every vertex-transitive graph $\xi = (v-1)(\bar{D}-1)$. This analogue of the HMS conjecture is known to be true in the special case of Cayley graphs (Theorem 3.6 of [13]). A well-known example of a vertex-transitive graph which is not in general a Cayley graph is the odd graph O_k [13]. Recall that the vertex set of O_k consists of all $k-1$-subsets of a $2k-1$-set. With these conventions, O_k has degree k and diameter $k-1$. Further the eccentricity

$\bar{D} = k - \frac{1}{2}$ [22]. By considering a set X such that $N(X)$ is the set of all points at distance $\lceil k/2 \rceil$ of a given point, we obtain for large k the estimate $\gamma = O(1/\sqrt{kv})$. By assuming the conclusion of the mentioned conjecture, we get $\gamma = \Omega(1/kv)$. Thus a better cut or a more subtle example has to be found.

3.5.2. Graphs on alphabets

It was shown in [7], using routings which consist neither uniquely of shortest paths nor uniquely "roundtrip" paths that, for the de Bruijn graph on d^D nodes,

$$\xi \leqslant v(D - 1).$$

From there we see that $\gamma \geqslant 2/(v(D - 1))$. Star cuts show that $\gamma \leqslant 2d/(v - 2d - 1)$. Analogous results hold for the Kautz graph [15].

3.5.3 Moore graphs

For the sake of simplicity we shall use the following (simplified) definition: a *Moore* graph is a regular graph of diameter 2 and girth 5. As is well-known, there are only two known such graphs: the Petersen graph of degree 3, and the Hoffman–Singleton graph of degree 7. Their spectra are computed on p. 165 of [8]. From Proposition 6.7.1 of [15] we known that then $\xi = \Delta(\Delta - 1)$. From there, we get $\gamma \geqslant 2/\Delta(\Delta - 1))$.

For instance, for the Petersen graph this yields $\gamma \geqslant \frac{1}{3}$ since $\xi = 6$ from [11]. In fact, it can be checked by a case by case analysis (either by hand or by computer) that $\gamma = \frac{1}{2}$. The eigenvalue bound only yields $\gamma \geqslant \frac{4}{15} \approx 0.27$. This is an instance where the combinatorial approach beats the eigenvalue method. For the Hoffman–Singleton graph $\xi = 42$, which entails $\gamma \geqslant \frac{1}{21} \approx 0.048$. This time, this is weaker than the eigenvalue estimate $\frac{4}{35} \approx 0.114$. It can be checked that this is also the case for the putative graph of degree 57. So, the Petersen graph is as exceptional as ever.

3.5.4. Miscellaenous

The *Heawood* graph is a cubic graph on 14 vertices whose picture is on [5, p. 236]. Its spectrum is item 4.19 of Table 4 of [8]. This yields $\gamma \geqslant 0.15$. Assuming that $\xi = (v - 1)(\bar{D} - 1)$, we get $\xi = 14$, yielding $\gamma \geqslant 0.142$ slightly less than the algebraic estimate. A computer search yields $\gamma = 0.25$ which corresponds to an optimal cut of size 4.

3.5.5. Platonic solids

In this subsection we investigate the skeleton of the five three-dimensional regular polytopes. The cube is treated in the elementary examples section, and γ is undefined for the tetrahedron. Pictures of the following three examples can be found on p. 234 of [5].

The *octahedron* is a regular graph of degree 4 on 6 vertices. Its spectrum is item 4.37 of Table 4 of [8]. The value of γ is visibly 4. Theorem 8 yields $\gamma \geqslant 0$, a poor estimate. Assuming $\xi = (v - 1)(\bar{D} - 1)$, which can be checked easily, we get $\xi = 1$, hence $\gamma \geqslant 2$, improving on the eigenvalue estimate.

The *icosahedron* graph is regular of degree 5 and has 12 vertices. It is number 4.19 of Table 4 of [8]. Theorem 8 yields $\gamma \geqslant 0.18$, when a computer search gives an actual value of $\frac{2}{3}$, achieved, by removing the hexagon between the two triangles on p. 236 of [5]. Assuming $\xi = (v-1)(\bar{D}-1)$, we get $\xi = 17$, hence $\gamma \geqslant 0.117$. This is even poorer than the linear algebra estimate.

The *dodecahedron* graph is cubic on 20 vertices. Its spectrum is number 4.49 of Table 4 of [8]. Theorem 8 yields $\gamma \geqslant 0.05$, when an obvious cut yields $\gamma \leqslant 0.1$, the true value (computer check). Assuming $\xi = (v-1)(\bar{D}-1)$, we get $\xi = 30$, hence $\gamma \geqslant 0.066\ldots$, a win of combinatorics over linear algebra.

4. Conclusion

As mentioned after Theorem 1, our bounds are optimistic for expanding parameters, but pessimistic for forwarding parameters. It would be interesting, for instance, to have an upper bound on π that would be a decreasing function of β. Also upper bounds on π as functions of the spectrum would be of interest. On the discussed examples spectral bounds are most effective for edge parameters and combinatorial techniques are roughly as good for vertex parameters.

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